

Exercises for 'Topics in complex analysis'

(05/11/2025)

**H 8.1 (Practice with Hadamard products)**

Find the Hadamard products for:

a)  $e^z - 1$ .

b)  $\cos(\pi z)$ .

**H 8.2 (Picard's little theorem for functions of finite order)**Show that if  $f$  is an entire function with finite order of growth that omits two values, then  $f$  is constant.**Remark:** We will soon prove this result for any entire function (Picard's little theorem).**Hint:** If  $f$  misses  $a$ , then  $f(z) - a$  is of the form  $e^{P(z)}$  where  $P$  is a polynomial.**H 8.3 (A transcendental equation)**Show that the equation  $e^z = z$  has infinitely many solutions  $z \in \mathbb{C}$ .**H 8.4 (Non-integral order of growth)**Show that if  $f$  is entire with finite order of growth  $\rho_f$  which is non-integral, then  $f$  has infinitely many zeros.**H 8.5 (Non-vanishing derivatives)**Suppose  $f$  is entire and never vanishes, and that none of its higher derivatives ever vanish. Prove that if  $f$  also has finite order of growth, then  $f(z) = e^{az+b}$  for some constants  $a, b \in \mathbb{C}$ .**H 8.6 (Order of growth in terms of Taylor coefficients)**Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  be an entire function of finite order of growth. Then the order of growth of  $f$  is intimately linked with the growth of the coefficients  $a_n$  as  $n \rightarrow \infty$ .a) Show that  $g(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha}$  describes an entire function of order  $1/\alpha$ , for any  $\alpha > 0$ .**Remark:** You may want to skip this item and return to it later, since it is a little complicated.b) Suppose  $|f(z)| \leq Ae^{a|z|^\rho}$  for every  $z \in \mathbb{C}$ , where  $\rho > 0$ . Then

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} n^{1/\rho} < \infty. \quad (1)$$

**Hint:** Use Cauchy's inequality

$$|a_n| \leq \sup_{|z|=r} \frac{|f(z)|}{r^n} \leq \frac{A}{r^n} e^{ar^\rho}$$

and the fact that the function  $u^{-n}e^{u^\rho}$ , for  $u > 0$ , attains its minimum value  $e^{n/\rho}(\rho/n)^{n/\rho}$  at  $u = n^{1/\rho}/\rho^{1/\rho}$ . Then choose  $r$  in terms of  $n$  to achieve this minimum.

c) Conversely, if (1) holds, then for any  $\varepsilon > 0$  we have  $|f(z)| \leq A_\varepsilon e^{a_\varepsilon |z|^{\rho+\varepsilon}}$  for every  $z \in \mathbb{C}$ .

**Hint:** Note that for  $|z| = r$ ,

$$|f(z) - a_0| \leq \sum_{n=1}^{\infty} \frac{c^n r^n}{n^{n/\rho}} \leq \sum_{n=1}^{\infty} \frac{c^n r^n}{(n!)^{1/\rho}}$$

for some constant  $c$ , since  $n^n \geq n!$ . This yields a reduction to item a).